Tensors and Interpolation

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Tensors appear in many problems in computer science and mathematics. The most famous family of tensors is certainly the family of matrices to which a huge amount of work has already been devoted. But there is nowadays an increasing interest in higher order tensors, since they show up in many applications: Statistics, Electrical engineering, Antenna Array Processing, Telecommunications, Chemometrics, Psychometrics, Data Analysis, Independent Component Analysis, Shape analysis, Complexity theory ... From a theoretical point of view, their study is also raising interesting and still unsolved problems connected to interpolation problems, Hilbert schemes, secant varieties, Gorenstein algebra, ... Tensor algebra plays an important role in modern algebraic topology, deformation theory, and the theory of operads.

The focus of this workshop is on higher order tensors such as symmetric tensors known as polynomials, antisymetric or tensors associated with general multilinear operators and its dual view via interpolation problems. In parallel the role of tensor algebras in algebraic topology will be explored, with an eye towards the interactions between these subjects.

More specifically but non-exhaustively, the topics include

- tensors analysis
- rank determination and decomposition,
- multivariate interpolation problems,
- approximation problems,
- geometric and algebraic properties of tensors,
- related algorithmic issues,
- applications related to higher order tensors.
- operads over monoidal categories,
- Batalin-Vilkovisky algebras and deformation theory,
- Koszul duality,
- definition of tensors over subfields,

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Secant varieties to tangential varieties of cubic Veronese embeddings.

Hirotachi Abo

Abstract. The past several decades have seen an interest in secant varieties cross an ever widening collection of desciplines including algebraic complexity theory, algebraic statistics, interpolation as well as algebraic geometry.

A well known classification of defective secant varieties of Veronese varieties has been completed in a series of papers by Alexander and Hirschowitz. There is a conjecturally completed list of defective secant varieties to tangential varieties of Veronese varieties suggested by Geramita, Catalisano and Gimigliano.

In this talk, I will discuss an inductive approach to establish the existence of large classes of non-defective secant varieties to tangential varieties of cubicc Veronese embeddings. This approach was inspired by the work of Brambilla and Ottaviani.

Sylvester's Algorithm

Alessendra Bernardi

Abstract. The Sylvester's algorithm allows to compute the symmetric rank of a symmetric tensor $T \in S^d V$ if V is a two dimensional vector space. In the first part of the talk I will present the versions of this that gives in a finite number of steps the symmetric rank of the tensor without giving an explicit decomposition of it. In the second part of the talk I will show how the same geometric idea can be applied to write effective algorithms also for some higher dimensional cases. What is presented in this talk comes from a work in progress with A. Gimigliano and M. Idà.

Decomposable Symmetric Tensors

Emmanuel Briand

Abstract. In the space of symmetric tensors $S^d V$, where V is a finite–dimensional vector space on $\mathbb{C}$, the decomposable tensors $f_1 f_2 \cdots f_d$ form an irreducible algebraic subvariety D. If V is the dual of a vector space W, then $S^d V$ identifies with the forms of degree d on W, and the decomposable tensors are just the products of linear forms.

An interesting problem is to find equations defing D in $S^d V$.

I will give an overview of the known solutions to this problem.

Some of these solutions are obtained through classical invariant theory. One such solution is due to Brill and Gordan, another one is due to Gaeta.

Another approach consists in working in coordinates. Then one meets objects that are interesting by themselves, the *diagonal invariants of the symmetric group*, that have very nice combinatorial properties that I will present.

Best subspace tensor approximations: a quantum physical interpretation

Patrick Cassam-Chenaï

Abstract. We will comment on a recent work by S. Friedland and V. Mehrmann (http://arxiv.org/abs/0805.4220), where the authors generalize the singular value decomposition method to obtain low-rank approximation of tensors. In particular, we will explain the underlying quantum physical concepts behind their proposed best subspace approximations as well as generalisations of these concepts to symmetric and exterior algebra elements. Illustrations will be taken from the field of quantum chemistry.

Some Numerical Results on the Rank of Generic Three-Way Arrays over R

Vartan Choulakian

Abstract. We introduce a new method for the numerical computation of the rank of a three-way array over R. We present some theoretical and numerical results based on the computation of Gröbner bases.

Key words: Tensors; three-way arrays; Candecomp/Parafac; generic rank; typical rank; Segre variety; Veronese variety; Gröbner bases.

Interpolation, residues and resultants

Alicia Dickenstein

Abstract. We will recall some classical results on the relation between interpolation modulo an affine complete intersection polynomial ideal and the Euler Jacobi vanishing conditions for global residues. We will show that the well known genericity assumptions for the validity of these results are in fact not only sufficient but also necessary. We will also recall the sparse case, together with some open questions.

Multivariate Hermite Interpolation : the state of affairs.

André Hirschowitz

Abstract. The problem of multivariate Hermite polynomial interpolation is still widely open, even in the bivariate case. We will review this problem together with connected questions and recent contributions.

Rank v.s. border rank

Joseph M. Landsberg

Abstract. The rank of a matrix admits numerous definitions, many of which fail to coincide when generalized to tensors (although all give rise to the same set of rank one tensors). The rank of a tensor T is defined to be the minimum number r of rank one elements a_i such that T=$a_1 + ... + a_r$. The set of tensors of rank at most r is not closed, and by definition the set of tensors of border rank at most r is the Zariski closure of this set. In this talk I will survey what is known about ranks and border ranks of tensors and symmetric tensors. While border rank is more natural from the perspective of algebraic geometry (and has been studied extensively), I'll show that the geometry assciated to rank also can be beautiful. Recent results in the talk are joint with Z. Teitler.

Pfaffians and Valiant's holographic algorithms

Joseph M. Landsberg

Abstract. If one computed the determinant of an n by n matirx using the naive formula, it would involve performing a number of arithmetic operations that is exponential in n. Gaussian elimination enables us to compute determinants in less than n^4 operations. There are many classes of problems and sequences of polynomials which naively require performing a number of arithmetic operations that is exponential in the input data to solve. Some, like the determinant, also admit fast algorithms. Valiant has obtained a class of such algorithms, which he calls "holographic" algorithms, which, from his perspective, allow one to simulate a quantum computer on a classical computer. I'll explain Valiant's algorithms, their potential applicability, how they relate to Pfaffians (as well as some identites regarding Pfaffians of interest in their own right), and what light they shed on the P vs NP problem.

From tensors to trees

Jean-Louis Loday

Abstract. Tensors appear naturally in the construction of the free associative algebra. We consider two variations of associative algebras: A-infinity algebras and dendriform algebras. In both cases appear the planar trees, but from a different perspective. In each case they are closely related to the Stasheff polytope (alias associahedron) and reveal its fantastic properties.

On the rank of skew-symmetric and general tensors.

Georgio Ottaviani

Abstract. The theorem of Alexander and Hirschowitz computes the generic rank of symmetric complex tensors. There is a general pattern, with a list of exceptions. We report about the analogous problem for skew-symmetric and general tensors. There are again some lists of exceptions to the general pattern, and conjectures that such lists are complete. This is joint work with H. Abo and C. Peterson.

The typical rank of real binary forms

Georgio Ottaviani

Abstract. A classical theorem of Sylvester computes the typical (generic) rank of complex binary forms. It is well known that for real cubic binary forms, there are two typical ranks. In a work in progress with P. Comon we show that for higher degree there are more than two typical ranks.

On Grothendieck—Serre's conjecture concerning principal G-bundles over a reductive group scheme

Ivan Panin

Abstract. Let R be a regular semi-local ring containing an infinite perfect subfield and let K be its field of fractions. Let G be a reductive R-group scheme satifying a mild "isotropy condition". Then each principal G-bundle P which becomes trivial over K is trivial itself. If R is of geometric type, then it suffices to assume that R is of geometric type over an infinite field.

Lie idempotents and decompositions in the tensor algebra.

Fréderic Patras

Abstract. A classical problem in the study of words (or tensors) is to provide effective formulas implementing the Poincaré-Birkhoff-Witt isomorphism. The canonical solution is given by the works of Solomon and Garsia-Reutenauer (from the late 60's to the mid-80's), featuring Solomon's idempotent, but further progress was done later (eg as far as other Lie idempotents are concerned). We will survey some classical features of the problem and some recent developments on the subject (including possibly joint works with Reutenauer, Schocker -links with the theory of species and operads- and some applications to quantum field theory -joint works with Ebrahimi-Fard, Gracia-Bondia, Manchon).

Complexité bilinéaire de la multiplication dans une extension finie d'un corps fini.

Robert Rolland & Stéphane Ballet

Abstract. La complexité bilinéaire de la multiplication dans une extension finie d'un corps fini est le rang du tenseur de la multiplication. Il a été démontré par des méthode d'interpolation sur des courbes algébriques que cette complexité est linéaire en le degré de l'extension, uniformément en la taille du corps fini de base. Nous proposons un survey sur les résultats que nous avons obtenus dans ce domaine.

Kroenecker coefficients

Mercedes Rosas

Abstract. Given two finite–dimensional vector spaces V and W on $\mathbb{C}$, the group $GL(V) \times GL(W)$ maps naturally to $GL(V \otimes W)$ (in coordinates, this is the "Kronecker product" of matrices). This gives to every $GL(V \otimes W)$–module a structure of $GL(V) \times GL(W)$–module. In particular, every irreducible representation of $GL(V \otimes W)$ decomposes as a sum of irreducible $GL(V) \times GL(W)$–modules. The multiplicities in these decompositions are the Kronecker coefficients.

Finding a nice combinatorial description of the Kronecker coefficients, akin to the Littlewood–Richardson rule for the Littlewood–Richardson coefficients, is a longstanding open problem in algebraic combinatorics. Recently, Mulmuley has set this problem at the heart of his *Geometric complexity theory*, a geoemtric approach to the algebraic analog of $P \neq NP$.

I will present recent results obtained with Rosa Orellana and Emmanuel Briand about a particular family of Kronecker coefficients: the *Kronecker coefficients indexed by three two–row shapes.* We obtained explicit formulas for these coefficients. These formulas are polynomial functions on $\mathbb{Z}^6$ (the indices of the considered Kronecker coefficients are points in this space), whose domains are the intersections $\sigma \cap C$ of the maximal cells σ of a fan of $\mathbb{R}^6$ with the cosets C of a full–rank sublattice of $\mathbb{Z}^6$.

I will explain how these formulas were obtained, and how they allowed us to disprove a conjecture, due to Mulmuley, about the behaviour of the stretching functions associated to the Kronecker coefficients.

A new algorithm for symmetric tensor decomposition

E. Tsigaridas

Abstract. We present an algorithm for decomposing a symmetric tensor, of dimension n and order d as a sum of rank-1 symmetric tensors, extending the algorithm of Sylvester devised in 1886 for binary forms.

We recall the correspondence between the decomposition of a homogeneous polynomial in n variables of total degree d as a sum of powers of linear forms (Waring's problem), incidence properties on secant varieties of the Veronese Variety and the representation of linear forms as a linear combination of evaluations at distinct points. Then we reformulate Sylvester's approach from the dual point of view.

Exploiting this duality, we propose necessary and sufficient conditions for the existence of such a decomposition of a given rank, using the properties of Hankel (and quasi-Hankel) matrices, derived from multivariate polynomials and normal form computations. This leads to the resolution of polynomial equations of small degree in non-generic cases. We exploit this characterization to develope a new algorithm for symmetric tensor decomposition, based on linear algebra computations with these Hankel matrices.

This is a joint work with J. Brachat, P. Comon and B. Mourrain.

Opérations sur la (co)homologie de Hochschild

Boris Tsygan

Abstract. Nous rapellerons les travaux sur les opérations sur les complexes de chaines et cochaines de Hochschild (Kontsevich-Soibelman, Tamarkin, Dolgushev-Tamarkin-Tsygan, Costello, van den Berg, Cattaneo-Felder-Willwacher et al.) Nous poserons quelques conjectures concernant les structures qui unissent ces opérations et les formules de Kunneth. Ces conjectures devraient contenir une description d'une structure algébrique sur le complex de déformation d'une algèbre de Hopf.

Efficient and Stable Decompositions for Tensors in Many Dimensions

Eugene Tyrtyshnikov

Abstract. Decompositions of d-dimensional tensors are crucial either in structure recovery problems and merely for a compact representation of tensors. However, the well-known decompositions have serious drawbacks: the Tucker decompositions suffer from exponential dependence on the dimensionality d while fixed-rank canonical decompositions are not stable.

In this talk we present new decompositions [1] that are stable and have the same number of representation parameters as canonical decompositions for the same tensor. Moreover, the new format

possesses nice stability properties of the SVD (as opposed to the canonical format) and is convenient for basic operations with tensors. This work is joint with Ivan Oseledets.

[1] I.V.Oseledets and E.E.Tyrtyshnikov, Breaking the curse of dimensionality, or how to use SVD in many dimensions, Research Report 09-03, Kowloon Tong, Hong Kong: ICM HKBU, 2009 (www.math.hkbu.edu.hk/ICM/pdf/09-03.pdf).

Propérade en Algèbre, Topologie, Géométrie et Physique Mathématique (HDR)

Bruno Vallette

Abstract. Une opérade est un objet algébrique qui sert à coder les opérations à plusieurs entrées et mais à une seule sortie, comme le produit d'une algèbre, et une propérade est un objet algébrique qui sert à coder les opérations à plusieurs entrées et à plusieurs sorties, comme le coproduit d'une cogèbre. Les opérades et les propérades sont des objets universels qui se retrouvent dans de nombreux domaines des mathématiques. J'illustrerai ceci avec des résultats obtenus en algèbre, topologie algébrique, géométrie et physique mathématique grace aux opérades et aux propérades.

Interpolation for generators and syzygies

Charles Walter

Abstract.