# IDENTIFIABILITY OF BPSK, MSK AND QPSK FIR SISO CHANNELS FROM MODIFIED SECOND-ORDER STATISTICS

*J-P. Delmas*(1), *P. Comon*(2), *Y. Meurisse*(1)

(1) INT, 9 rue Charles Fourier, F-91011 Evry cedex, France
jean-pierre.delmas(yann.meurisse)@int-evry.fr
(2) I3S, CNRS, 2000 route des Lucioles, F-06903 Sophia-Antipolis cedex, France
p.comon@ieee.org

## ABSTRACT

This paper considers the problem of blind estimation of Finite Impulse Responses (FIR) of Single-Input Single-Output (SISO) channels from second order statistics of transformed data, when the channel is excited by Binary Phase Shift Keying (BPSK), Minimum Shift Keying (MSK) or Quadrature Phase Shift Keying (QPSK) inputs. Identifiability conditions are derived by considering that noncircularity induces diversity. Performance issues are also addressed by using standard subspace-based estimators, with benchmarks such as asymptotically minimum variance (AMV) bounds based on different statistics.

## 1. INTRODUCTION

SISO blind identification has been long considered to need High-Order Statistics (HOS) [1]. On the other hand, the use of additional diversity at the receiver permitted to build a SIMO channel that could be identified with the sole help of second order statistics, e.g. via subspace techniques [2]; if spatial diversity is not available at the receiver, oversampling allows to increase diversity only in the presence of sufficient excess bandwidth, which is however rarely encountered. Other more recent techniques incorporate the knowledge of the symbol constellation, which eventually amounts to using noncircularity of the symbol sequence (see e.g., [3]).

In this paper, second or higher order noncircularity is utilized in order to restore identifiability of FIR SISO channels in the absence of space or bandwidth diversities. This results in simple SIMO-type Blind Identification algorithms based on second order statistics of *transformed data*; these transformations include complex conjugation, but also monomial functions.

This paper is organized as follows. Section 2 introduces FIR SISO data models. Identifiability results are given in section 3, and performance issues are addressed in section 4. Some illustrative examples are eventually reported in section 5.

## 2. DATA MODEL

Limiting our discussion to linear modulations, the complex envelope of a transmitted signal $s(t)$ takes the baseband expression $s(t) = \sum_k g(t-kT)\, x_k$, where $x_k$ denotes the discrete sequence of transmitted symbols, $T$ the symbol period, and $g(\cdot)$ the transmit filter. After propagation through a time-invariant channel, the signal received on the antenna is of the form $s(t) = \sum_k h(t-kT)\, x_k$, for some complex linear filter $h(\cdot)$ representing the global channel, combining transmit and receive filters with the channel. It is subsequently assumed that the global channel can be approximated by a FIR filter. Thus, if sampled exactly at the rate $1/T$, the received discrete-time signal may be modelled as

$$y_t = \sum_{k=0}^{M} h_k\, x_{t-k} + n_t$$

where $h_k = h(kT)$, $0 \le k \le M$, and $n_t$ denotes an additive noise, which is assumed to be second-order circular (i.e. $\mathrm{E}(n_t^2) = 0$), independently identically distributed (i.i.d.), zero-mean and with finite variance $\sigma_n^2 = \mathrm{E}|n_t|^2$. The information symbol sequence, $x_t$, and noise $n_t$, are assumed to be statistically independent; $x_t$ can be a stationary or a cyclo-stationary process. It is convenient to use a multivariate representation by stacking $M+1$ samples of the received signal:

$$\mathbf{y}_t = (y_t, y_{t-1}, ..., y_{t-M})^T = \mathcal{H}(\mathbf{h})\mathbf{x}_t + \mathbf{n}_t \quad (2.1)$$

with $\mathbf{x}_t = (x_t, x_{t-1}, ..., x_{t-2M})^T$ and $\mathbf{n}_t =$

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$(n_t, n_{t-1}, ..., n_{t-M})^T$, and where $\mathcal{H}(\mathbf{h})$ is the following $(M+1) \times (2M+1)$ Töplitz matrix:

$$\mathcal{H}(\mathbf{h}) = \begin{pmatrix} h_0 & \cdots & \cdots & h_M & & \\ & \ddots & & & \ddots & \\ & & h_0 & \cdots & \cdots & h_M \end{pmatrix}$$

with $\mathbf{h} \stackrel{\text{def}}{=} (h_0, h_1, ..., h_M)^T$. In the following, the cases of BPSK, MSK and QPSK modulations will be considered, as working examples.

## 3. IDENTIFIABILITY

### 3.1. BPSK modulation

In this section, $x_t$ is a stationary process, possibly colored, taking its values in the set $\{-1, +1\}$ with equal probabilities. It is assumed that the so-called *noncircular* covariance $\mathbf{R}_x \stackrel{\text{def}}{=} \mathrm{E}(\mathbf{x}_t \mathbf{x}_t^T)$ is non singular. The set of nonzero circular and noncircular second-order statistics of $y_t$ can be gathered in the covariance matrix of the extended vector $\tilde{\mathbf{y}}_t = (\mathbf{y}_t^T, \mathbf{y}_t^H)^T$, so that from (2.1):

$$\mathbf{R}_{\tilde{y}} \stackrel{\text{def}}{=} \mathrm{E}(\tilde{\mathbf{y}}_t \tilde{\mathbf{y}}_t^H) = \qquad (3.1)$$
$$\begin{bmatrix} \mathcal{H}(\mathbf{h}) \\ \mathcal{H}(\mathbf{h}^*) \end{bmatrix} \mathbf{R}_x \begin{bmatrix} \mathcal{H}(\mathbf{h})^H & \mathcal{H}(\mathbf{h}^*)^H \end{bmatrix} + \sigma_n^2 \mathbf{I}_{2M+2}$$

Consequently, we obtain this way a structured covariance matrix similar to that obtained in the SIMO case; here the two channels have impulse responses $\mathbf{h}$ and $\mathbf{h}^*$. Therefore the results (see e.g., [4]) concerning the identifiability of SIMO channels can be applied. Because $2(M+1) > 2M+1$ and $\mathbf{R}_x$ is nonsingular, the range space of the filtering matrix $\tilde{\mathcal{H}}(\mathbf{h}) \stackrel{\text{def}}{=} \begin{bmatrix} \mathcal{H}(\mathbf{h}) \\ \mathcal{H}(\mathbf{h}^*) \end{bmatrix}$ is identifiable from $\mathbf{R}_{\tilde{y}}$, and this range space determines the channel coefficients up to a multiplicative constant if channels $\mathbf{h}$ and $\mathbf{h}^*$ do not share any common zeros. This ambiguity can be fixed by using the knowledge of the alphabet; we have proved the following

**Result 1** *With a BPSK modulation and additive noise satisfying the above assumptions, the impulse response* $\mathbf{h}$ *of a SISO channel is identifiable from the circular and noncircular second-order statistics of its output if the polynomial* $\sum_{k=0}^{M} h_k z^k$ *has neither real zeros nor conjugated zeros.*

### 3.2. MSK modulation

Now, we suppose $x_t$ is a MSK modulation defined by $x_{t+1} \stackrel{\text{def}}{=} i x_t c_t$ where $c_t$ is a sequence of independent BPSK symbols $\{-1, +1\}$ with equal probabilities where the original value $x_0$ remains unspecified in the set $\{+1, +i, -1, -i\}$. This process may be equivalently modeled (see e.g., [5]) as $x_t = i^t b_t x_0$ where $b_t$ is another sequence of independent BPSK symbols $\{-1, +1\}$ with equal probabilities.

We note that $x_t$ (and thus $y_t$) is not stationary. But by down-modulating[1] each $y_t$, we get

$$y'_t \stackrel{\text{def}}{=} y_t i^{-t} = \sum_{k=0}^{M} h'_k b_{t-k} + n'_t,$$

with $h'_k \stackrel{\text{def}}{=} x_0 h_k i^{-k}, k = 0, ..., M$ and where $n'_t \stackrel{\text{def}}{=} i^{-t} n_t$ is still second order stationary. Gathering again the set of nonzero circular and noncircular second-order statistics of $y'_t$ in the covariance matrix of the extended vector $\tilde{\mathbf{y}}'_t = (\mathbf{y'}_t^T, \mathbf{y'}_t^H)^T$ we obtain

$$\mathbf{R}_{\tilde{y}'} \stackrel{\text{def}}{=} \mathrm{E}(\tilde{\mathbf{y}}'_t \tilde{\mathbf{y}}'^H_t) = \qquad (3.2)$$
$$\begin{bmatrix} \mathcal{H}(\mathbf{h}') \\ \mathcal{H}(\mathbf{h}'^*) \end{bmatrix} \begin{bmatrix} \mathcal{H}(\mathbf{h}')^H & \mathcal{H}(\mathbf{h}'^*)^H \end{bmatrix} + \sigma_n^2 \mathbf{I}_{2M+2}.$$

Consequently, we can use the same approach as for the BPSK modulation. Because $\sum_{k=0}^{M} h'_k z^k = x_0 \sum_{k=0}^{M} h_k (i^{-1} z)^k$, we obtain the following result

**Result 2** *With a MSK modulation and additive noise satisfying the above assumptions, the impulse response* $\mathbf{h}$ *of a SISO channel is identifiable from the circular and noncircular second-order statistics of its down-modulated output if the polynomial* $\sum_{k=0}^{M} h_k z^k$ *has neither purely imaginary zeros nor paired zeros of the form* $(iz_0, iz_0^*)$, $z_0 \in \mathbb{C}$.

### 3.3. QPSK modulation

Now, we suppose $x_t$ is a QPSK modulation defined as a sequence of i.i.d. r.v. taking their values in the set $\{+1, +i, -1, -i\}$ with equal probabilities and now $n_t$ is Gaussian distributed and circular. Consequently $y_t$ is now second-order circular. However, noting that $x_t^2$ is a BPSK modulation, a similar approach can still be used by squaring the outputs $y_t$. Indeed define

[1] In [3], this down-modulation is performed on vector $\mathbf{y}_t$ and consequently $\mathbf{y}'_t = \mathbf{y}_t i^{-t}$ becomes stationary at the second order but not $y'_t$.

$y'_t \stackrel{\text{def}}{=} y_t^2$ and $h'_k \stackrel{\text{def}}{=} h_k^2, k = 0, ..., M$. Then, by using (i) the multi-linearity of moments and cumulants [6] [7], (ii) relations between moments and cumulants, (iii) properties specific to the QPSK alphabet, namely $x_t^4 = 1$, $\mathrm{E}(x_t^2) = 0$ and $\mathrm{E}|x_t^2| = 1$, and (iv) the whiteness of $x_t$ at order 4, we can obtain the following second-order statistics of the modified input, $y'_t$:

$$r_{y'}(\ell) \stackrel{\text{def}}{=} -\mathrm{E}(y'_t y'^*_{t-\ell}) + 2\left[\mathrm{E}(y_t y^*_{t-\ell})\right]^2 = \sum_{k=\ell}^{M} h'_k h'^*_{k-\ell}$$

$$r'_{y'}(\ell) \stackrel{\text{def}}{=} \mathrm{E}(y'_t y'_{t-\ell}) = \sum_{k=\ell}^{M} h'_k h'_{k-\ell}$$

for $\ell = 0, ..., M$. Gathering the second-order statistics $r'_{y'}(l)$ and $r_{y'}(l)$ in an Hermitian $2 \times 2$ block Töplitz matrix, we obtain the extended covariance matrix

$$\mathbf{R}_{\tilde{y}'} = \begin{bmatrix} \mathcal{H}(\mathbf{h}') \\ \mathcal{H}(\mathbf{h}'^*) \end{bmatrix} \begin{bmatrix} \mathcal{H}(\mathbf{h}')^H & \mathcal{H}(\mathbf{h}'^*)^H \end{bmatrix}. \quad (3.3)$$

As a consequence, a structured covariance matrix has been obtained, which is similar to that obtained in the SIMO case, with two channels of impulse responses $\mathbf{h}'$ and $\mathbf{h}'^*$. Therefore the results (see e.g., [4]) concerning the identifiability of SIMO channels can be applied as well. The range space of $\mathbf{R}_{\tilde{y}'}$ determines the channel coefficients of $\mathbf{h}'$ and $\mathbf{h}'^*$ up to a multiplicative constant if these channels do not share any common zeros. This ambiguity can be reduced by using the knowledge of the symbol alphabet. Returning to $\mathbf{h}$, each coefficient of $\mathbf{h}$ is determined up to a sign ambiguity, which may be cleared up using the successive second-order statistics $\mathrm{E}(y_t y^*_{t-l}), l = M, M-1, ..., 1$. Consequently, we obtain the following result:

**Result 3** *With a QPSK modulation and additive noise satisfying the above assumptions, the impulse response* $\mathbf{h}$ *of a channel is identifiable from the second-order statistics of the squared output if the polynomial* $\sum_{k=0}^{M} h_k^2 z^k$ *has neither real zeros nor conjugated zeros.*

## 4. PERFORMANCE

The above identifiability results naturally raise the important issue of performance analysis of algorithms based on the second-order covariance $\mathbf{R}_{\tilde{y}}$, and on the modified second-order statistics $\mathbf{R}_{\tilde{y}'}$. Usually, the Cramer-Rao bound (CRB) serves as a useful benchmark for unbiased estimators yielded by identification algorithms. Because this CRB appears to be computationally prohibitive in the present context, the notion of asymptotically (in the number of measurements) minimum variance (AMV) bound introduced by Porat and Friedlander [8] and Stoica *et al* [9] is considered.

### 4.1. AMV bounds based on extended covariance matrices

Let $\boldsymbol{\alpha} = (\boldsymbol{\theta}^T, \boldsymbol{\rho}^T)^T$ denote the real-valued unknown parameters (containing the real and imaginary parts of the complex parameters) of the extended covariance matrices $\mathbf{R}_{\tilde{y}}$ or $\mathbf{R}_{\tilde{y}'}$, where $\boldsymbol{\theta} \stackrel{\text{def}}{=} (\Re(h_0), ..., \Re(h_M), Im(h_0), ..., Im(h_M))^T$, and where $\boldsymbol{\rho}$ collects the nuisance parameters for the BPSK and MSK modulations. Depending on the a priori knowledge of the inputs, $\boldsymbol{\rho} = \sigma_n^2$ if the BPSK sequence $x_t$ or the driving sequence $b_t$ for the MSK modulation are white, and $\boldsymbol{\rho} = (\mathrm{E}(x_t x_{t-1}), ..., \mathrm{E}(x_t x_{t-2M}), \sigma_n^2)^T$ for correlated BPSK symbols.

We note that if the conditions of the results of Section 3 are satisfied, $\boldsymbol{\alpha}$ is identifiable from $\mathbf{R}_{\tilde{y}}$ or $\mathbf{R}_{\tilde{y}'}$, except the intrinsic ambiguity in the identification of $\mathbf{h}$, viz: a sign ambiguity for BPSK, or a rotation of $\pi/2$ ambiguity for MSK and QPSK modulations. The sign ambiguity of each $h(k)$ appearing in the case of the QPSK modulation can indeed be fixed by resorting to $\mathrm{E}(y_t y^*_{t-l})$.

These block Töplitz matrices $\mathbf{R}_{\tilde{y}}$ and $\mathbf{R}_{\tilde{y}'}$ are traditionally estimated from $T$ successive received signals $\mathbf{y}_t$ by replacing the various expectations by the associated sample correlations: $r_{y,T}(\ell) \stackrel{\text{def}}{=} \frac{1}{T}\sum_{t=1}^{T} y_t y^*_{t-\ell}$, $r'_{y,T}(\ell) \stackrel{\text{def}}{=} \frac{1}{T}\sum_{t=1}^{T} y_t y_{t-\ell}$, $r_{y',T}(\ell) \stackrel{\text{def}}{=} \frac{1}{T}\sum_{t=1}^{T} y'_t y'^*_{t-\ell}$ and $r'_{y',T}(\ell) \stackrel{\text{def}}{=} \frac{1}{T}\sum_{t=1}^{T} y'_t y'_{t-\ell}$. In order to apply the AMV bound [8, 9] to an arbitrary consistent extended second-order algorithm based on the sample estimates $\mathbf{R}_{\tilde{y},T}$ or $\mathbf{R}_{\tilde{y}',T}$ of $\mathbf{R}_{\tilde{y}}$ or $\mathbf{R}_{\tilde{y}'}$ respectively, the involved statistics $\mathbf{s}_T$ must collect real-valued sample correlations and complex-valued sample correlations and their conjugate [10]. For example, $\mathbf{s}_T = (r_{y,T}(0), r_{y,T}(1), ..., r_{y,T}(M), r^*_{y,T}(1), ..., r^*_{y,T}(M), r'_{y,T}(0), ..., r'_{y,T}(M), r'^*_{y,T}(0), ..., r'^*_{y,T}(M))^T$ for BPSK modulations.

In these conditions, the asymptotic covariance $\mathbf{C}_{\boldsymbol{\alpha}}$ of an estimator of $\boldsymbol{\alpha}$ given by an arbitrary consistent second-order algorithm based on these statistics $\mathbf{s}_T$ is bounded below by the real symmetric positive definite matrix $[\mathbf{S}^H(\boldsymbol{\alpha})\mathbf{C}_s^{-1}(\boldsymbol{\alpha})\mathbf{S}(\boldsymbol{\alpha})]^{-1}$ where $\mathbf{S}(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} \frac{d\mathbf{s}(\boldsymbol{\alpha})}{d\boldsymbol{\alpha}}$

with $\mathbf{s}(\boldsymbol{\alpha}) \stackrel{\text{def}}{=} \mathrm{E}(\mathbf{s}_T)$ and where $\mathbf{C}_s(\boldsymbol{\alpha})$ is the circular covariance of the asymptotic distribution of $\mathbf{s}_T$ [10]. Furthermore, there exists a nonlinear least square algorithm (dubbed the AMV algorithm [8]) whose covariance of the asymptotic distribution of the estimate of $\boldsymbol{\alpha}$ attains this lower bound.

For the BPSK modulation with correlated symbols, $\mathbf{s}(\boldsymbol{\alpha})$ is structured as $\mathbf{s}(\boldsymbol{\alpha}) = \boldsymbol{\Psi}(\boldsymbol{\theta})\boldsymbol{\rho} + \boldsymbol{\psi}(\boldsymbol{\theta})$, which implies $\mathbf{S} = [\mathbf{S}_1, \boldsymbol{\Psi}]$ with $\mathbf{S}_1 \stackrel{\text{def}}{=} \frac{\partial \mathbf{s}(\boldsymbol{\alpha})}{\partial \boldsymbol{\theta}}$ and the matrix inversion lemma gives

$$\mathbf{C}_{\boldsymbol{\theta}} \geq \left(\mathbf{S}_1^T \mathbf{C}_s^{-1/2} \boldsymbol{\Pi}^{\perp}_{\mathbf{C}_s^{-1/2}\boldsymbol{\Psi}} \mathbf{C}_s^{-1/2} \mathbf{S}_1\right)^{-1} \tag{4.1}$$

where $\boldsymbol{\Pi}^{\perp}_{\mathbf{A}}$ denotes the projector onto the orthogonal complement of the columns of $\mathbf{A}$. For the BPSK and MSK with the a priori knowledge that $\sigma_n^2$ is the unique nuisance parameter, $\mathbf{s}(\boldsymbol{\alpha})$ is structured as $\mathbf{s}(\boldsymbol{\alpha}) = \boldsymbol{\Phi}(\boldsymbol{\theta}) + \sigma_n^2 \mathbf{e}_1$, where $\mathbf{e}_1$ is the first unit vector in $\mathbb{R}^{4M+3}$. This implies $\mathbf{S} = [\mathbf{S}_1, \mathbf{e}_1]$, and the matrix inversion lemma yields this time:

$$\mathbf{C}_{\boldsymbol{\theta}} \geq \left(\mathbf{S}_1^T \mathbf{C}_s^{-1/2} \boldsymbol{\Pi}^{\perp}_{\mathbf{C}_s^{-1/2}\mathbf{e}_1} \mathbf{C}_s^{-1/2} \mathbf{S}_1\right)^{-1} \tag{4.2}$$

Lastly for the QPSK modulation, where there is no nuisance parameter in $\mathbf{R}_{\tilde{y}'}$, we obtain the lower bound

$$\mathbf{C}_{\boldsymbol{\theta}} \geq [\mathbf{S}^H(\boldsymbol{\theta}) \mathbf{C}_s^{-1}(\boldsymbol{\theta}) \mathbf{S}(\boldsymbol{\theta})]^{-1}. \tag{4.3}$$

Note that the expression of $\mathbf{C}_s$ for the different statistics involved are deduced from Bartlett formulas for the second-order and from extended Bartlett formulas for the fourth-order [11].

### 4.2. AMV bounds based on orthogonal projectors

In order to assess the performance of subspace-based algorithms built from $\mathbf{R}_{\tilde{y},T}$ or $\mathbf{R}_{\tilde{y}',T}$, it is relevant to replace the previous statistics $\mathbf{s}_T$ by the orthogonal projectors $\boldsymbol{\Pi}_{\tilde{y},T}$ or $\boldsymbol{\Pi}_{\tilde{y}',T}$ onto the noise subspace of $\mathbf{R}_{\tilde{y},T}$ or $\mathbf{R}_{\tilde{y}',T}$ respectively. In this case, $\mathbf{s}_T = \mathrm{vec}(\boldsymbol{\Pi}_{\tilde{y},T})$ or $\mathbf{s}_T = \mathrm{vec}(\boldsymbol{\Pi}_{\tilde{y}',T})$, and $\mathrm{E}(\mathbf{s}_T)$ depends only on $\boldsymbol{\theta}$. But $\boldsymbol{\theta}$ is identifiable from $\boldsymbol{\Pi}_{\tilde{y}}$ or $\boldsymbol{\Pi}_{\tilde{y}'}$ up to a complex scalar multiplication only. To avoid this ambiguity, one parameter of $\mathbf{h}$ is fixed to a predefined value, say, $h_0 = 1$. This choice will be also applied for the estimation procedures in the following subsection. In this case, the circular covariances of the asymptotic distribution of $\mathrm{vec}(\boldsymbol{\Pi}_{\tilde{y},T})$ and $\mathrm{vec}(\boldsymbol{\Pi}_{\tilde{y}',T})$ are singular, but it is proved in [12] that the AMV bound definition can be extended and is given by

$$\mathbf{C}_{\boldsymbol{\theta}} \geq [\mathbf{S}^H(\boldsymbol{\theta}) \mathbf{C}_s^{\#}(\boldsymbol{\theta}) \mathbf{S}(\boldsymbol{\theta})]^{-1} \tag{4.4}$$

where $\#$ denotes the Moore-Penrose inverse, and where here $\mathbf{S}(\boldsymbol{\theta}) \stackrel{\text{def}}{=} \frac{d\mathbf{s}(\boldsymbol{\theta})}{d\boldsymbol{\theta}}$ with $\mathbf{s}(\boldsymbol{\theta}) = \mathrm{vec}(\boldsymbol{\Pi}_{\tilde{y}})$ or $\mathbf{s}(\boldsymbol{\theta}) = \mathrm{vec}(\boldsymbol{\Pi}_{\tilde{y}'})$ and where closed-form expressions of $\mathbf{C}_s(\boldsymbol{\theta})$ are given in [12].

**Remark 1** Note that these expressions of $\mathbf{C}_s(\boldsymbol{\theta})$ depend only on the statistics of the inputs $x_t$ through the second-order terms for BPSK and MSK symbols and the second and fourth-order terms for QPSK symbols. On the other hand, the expressions of $\mathbf{C}_s(\boldsymbol{\theta})$ associated with the statistics $\mathbf{R}_{\tilde{y},T}$ and $\mathbf{R}_{\tilde{y}',T}$ depend on the second and fourth-order terms for BPSK and MSK symbols, and on the second, fourth, sixth and eighth-order terms for QPSK symbols.

**Remark 2** Note that the lower bounds (4.1), (4.2) and (4.3) denoted $\mathbf{C}_{\boldsymbol{\theta}}^{\mathrm{AMV}(\mathbf{R}_{\tilde{y}})}$ or $\mathbf{C}_{\boldsymbol{\theta}}^{\mathrm{AMV}(\mathbf{R}_{\tilde{y}'})}$ associated with arbitrary consistent algorithms based on $\mathbf{R}_{\tilde{y}}$ or $\mathbf{R}_{\tilde{y}'}$ satisfy, for the same a priori knowledge:

$$\begin{aligned} \mathbf{C}_{\boldsymbol{\theta}}^{\mathrm{AMV}(\mathbf{R}_{\tilde{y}})} &\leq \mathbf{C}_{\boldsymbol{\theta}}^{\mathrm{AMV}(\boldsymbol{\Pi}_{\tilde{y}})} \\ \mathbf{C}_{\boldsymbol{\theta}}^{\mathrm{AMV}(\mathbf{R}_{\tilde{y}'})} &\leq \mathbf{C}_{\boldsymbol{\theta}}^{\mathrm{AMV}(\boldsymbol{\Pi}_{\tilde{y}'})} \end{aligned}$$

where $\mathbf{C}_{\boldsymbol{\theta}}^{\mathrm{AMV}(\boldsymbol{\Pi}_{\tilde{y}})}$ and $\mathbf{C}_{\boldsymbol{\theta}}^{\mathrm{AMV}(\boldsymbol{\Pi}_{\tilde{y}'})}$ denote the lower bound (4.4) associated with $\boldsymbol{\Pi}_{\tilde{y},T}$ and $\boldsymbol{\Pi}_{\tilde{y}',T}$ respectively, because these statistics are functions of $\mathbf{R}_{\tilde{y},T}$ and $\mathbf{R}_{\tilde{y}',T}$ respectively. Furthermore, it is proved in [13] that the lower bounds $\mathbf{C}_{\boldsymbol{\theta}}^{\mathrm{AMV}(\mathbf{R}_{\tilde{y}})}$ and $\mathbf{C}_{\boldsymbol{\theta}}^{\mathrm{AMV}(\boldsymbol{\Pi}_{\tilde{y}})}$ coincide for correlated BPSK symbols, and for uncorrelated BPSK symbols if this a priori knowledge is not taken into account.

### 4.3. Subspace-based algorithms

Because the structure of the covariance matrices $\mathbf{R}_{\tilde{y}}$ and $\mathbf{R}_{\tilde{y}'}$ of (3.1) (3.2) and (3.3) are similar to those obtained in the SIMO case, all the algorithms devised in this case can be used in the present context. In the sequel by lack of place, only the so-called least square (LS) and subspace (SS) will be considered [14]. We note that the LS and SS estimates coincide in the two-channel case [14] when the noise subspace is of one dimension, which is satisfied for BPSK and MSK modulations. In this case[2], any eigenvector $\tilde{\mathbf{v}}$ associated with the smallest eigenvalue of the block structured matrices $\mathbf{R}_{\tilde{y}}$ satisfies $\tilde{\mathbf{v}} = (\mathbf{v}_1^T, \mathbf{v}_1^H e^{i\phi})^T$ and:

$$\begin{bmatrix} \mathbf{h} \\ \mathbf{h}^* \end{bmatrix} = c \begin{bmatrix} \mathbf{v}_1 \\ -\mathbf{v}_1^* e^{i\phi} \end{bmatrix}.$$

[2] We only consider the BPSK modulation, the case of the MSK modulation is similar.

If from now, $\mathbf{v}_1$ is constrained to have its first component to be unity: $\mathbf{h} = \mathbf{v}_1$ ($c = 1$, $\phi = \pi$). Consequently, the LS and SS estimates are given by $\mathbf{h}_T = \mathbf{v}_{1,T}$ where $\tilde{\mathbf{v}}_T = (\mathbf{v}_{1,T}^T, \mathbf{v}_{1,T}^H e^{i\phi_T})^T$ denotes the eigenvector associated with the smallest eigenvalue of the block structured matrices $\mathbf{R}_{\tilde{y},T}$ whose first component is unity.

The asymptotic performance of this algorithm can be devised from the asymptotic distribution of $\tilde{\mathbf{v}}_T$:

$$\sqrt{T}(\tilde{\mathbf{v}}_T - \tilde{\mathbf{v}}) \xrightarrow{\mathcal{L}} \mathcal{N}\left(\mathbf{0}, \mathbf{C}_{\tilde{v}}, \mathbf{C}'_{\tilde{v}}\right),$$

where $\mathbf{C}_{\tilde{v}}$ and $\mathbf{C}'_{\tilde{v}}$ are derived from the mapping $\mathbf{R}_{\tilde{y},T} \longmapsto \tilde{\mathbf{u}}_T \longmapsto \tilde{\mathbf{v}}_T$ where $\tilde{\mathbf{u}}_T$ denotes the unit norm eigenvector associated with the smallest eigenvalue of $\mathbf{R}_{\tilde{y},T}$ with positive real first component. We obtain by the chain rule in particular:

$$\mathbf{C}_{\tilde{v}} = \mathbf{D}(\tilde{\mathbf{v}}^T \otimes \mathbf{S}^{\#})\mathbf{C}_{R_{\tilde{y}}}(\tilde{\mathbf{v}}^* \otimes \mathbf{S}^{\#})\mathbf{D}^H$$

where $\mathbf{S} \stackrel{\text{def}}{=} \mathbf{R}_{\tilde{y}} - \sigma_n^2\mathbf{I}$ with $\mathbf{D} \stackrel{\text{def}}{=} \frac{d\tilde{\mathbf{v}}}{d\tilde{\mathbf{u}}} = \sqrt{2}\|\mathbf{h}\|\left(\mathbf{I} - \sqrt{2}\|\mathbf{h}\|(\tilde{\mathbf{v}}, \mathbf{O})\right)$. After some algebraic manipulations we obtain:

$$\mathbf{C}_{\tilde{v}} = \mathbf{D}\mathcal{S}^{\#}\left(\sigma_n^2\mathbf{R}_{\tilde{y}} + \mathbf{R}'_{\tilde{y}}\mathbf{\Pi}^*_{\tilde{y}}\mathbf{R}'^*_{\tilde{y}}\right)\mathcal{S}^{\#}\mathbf{D}^H \qquad (4.5)$$

where $\mathbf{R}'_{\tilde{y}} \stackrel{\text{def}}{=} \mathrm{E}(\tilde{\mathbf{y}}_t\tilde{\mathbf{y}}_t^T) = \tilde{\mathcal{H}}(\mathbf{h})\mathbf{R}_x\tilde{\mathcal{H}}^T(\mathbf{h}) + \sigma_n^2\mathbf{J}$ and with $\mathbf{J} \stackrel{\text{def}}{=} \begin{pmatrix} \mathbf{O} & \mathbf{I} \\ \mathbf{I} & \mathbf{O} \end{pmatrix}$.

## 5. ILLUSTRATIVE EXAMPLES

Fig.1 exhibits the asymptotic lower bound MSE $(\mathbf{h}) = \frac{1}{T}\mathrm{Tr}(\mathbf{C}_{\boldsymbol{\theta}}^{\mathrm{AMV}(\mathbf{\Pi}_{\tilde{y}})})$ and the asymptotic theoretical and actual MSE given by the LS/SS algorithm for independent BPSK symbols or MSK symbols driven by independent symbols with the channel $h(z) = (1 - z_1^{-1}z)(1 - z_2^{-1}z)$ with $z_1 = 0.8e^{i\alpha}$ and $z_2 = 1.25e^{i\pi/4}$ where $\alpha$ varies from 0 to $2\pi$ for $T = 1000$ and $SNR = 20$dB.

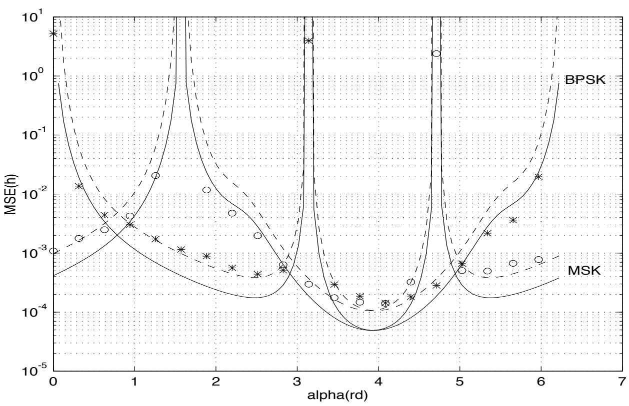


Fig.1 For either BPSK or MSK inputs: (—) Asymptotic lower bound, (- -) asymptotic theoretical MSE($\mathbf{h}$), and (o, $\star$) actual MSE($\mathbf{h}$), given by the LS/SS algorithm.

We note that these MSE increase dramatically when the zero $z_1$ approaches the real [resp. imaginary] axis for the BPSK [resp. MSK] modulation for which $\mathbf{h}$ becomes nonidentifiable. This behavior is explained by the behavior of the pseudoinverses $\mathbf{C}_s^{\#}(\boldsymbol{\theta})$ and $\mathcal{S}^{\#}$ in (4.4) and (4.5) respectively, for which $\tilde{\mathcal{H}}(\mathbf{h})$ becomes rank deficient.

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